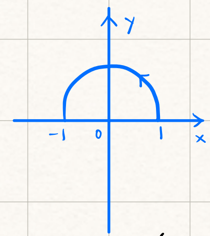
1. Let c be the semicircular arc from 1 to -1 in the upper half plane. Prove that :

$$\left| \int_C \frac{e^z}{z}\, dz \right| \le \pi e$$



2. Proof of Cauchy – Goursat Theorem. (on page 150)

without Green's THM
drop "C'" assumption.

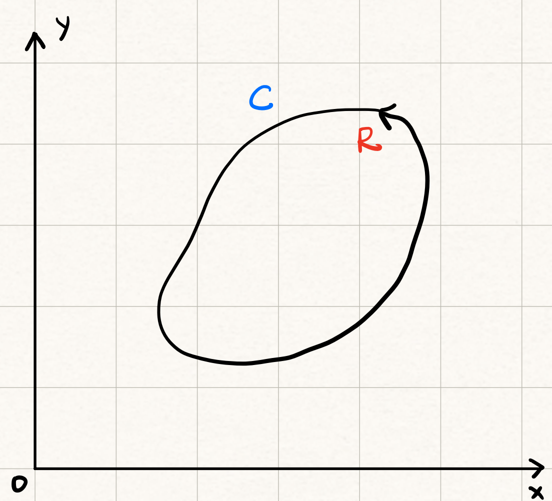
THM (Cauchy – Goursat Theorem.)

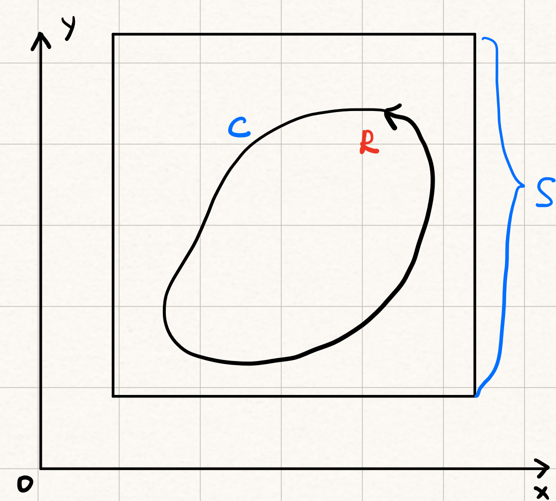
If f is holomorphic at all points interior to and on a simple closed path C, then

(i) $\int_C f(z)\, dz = 0$, $\int_{\tilde{C}} f(z) = 0$ for all closed path $\tilde{C} \subset D$

(ii) f is path independent in D

(iii) f has an antiderivative in D.





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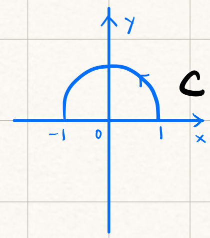


Sol: $\left| \int_c \frac{e^z}{z}\, dz \right| \le \max\limits_{z \in C} \left|\frac{e^z}{z}\right| \cdot L(C)$

sec. 47.

$$L(C) = \frac{2\pi}{2} = \pi .$$

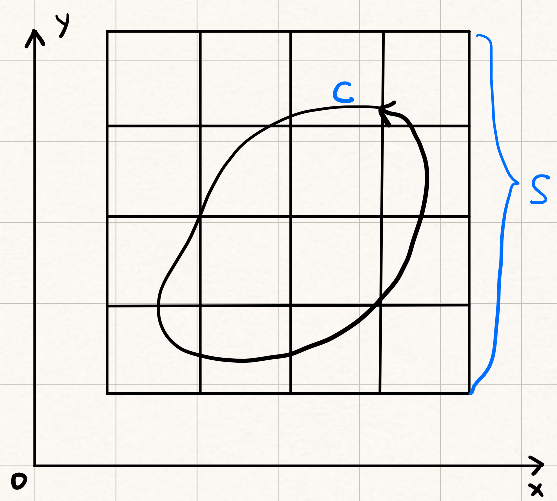
$$\max_{z \in C} \left|\frac{e^z}{z}\right| = \max_{z \in C} \frac{|e^z|}{|z|} \underset{|z|=1}{\overset{z=e^{i\theta}}{=}} \max_{z \in C} |e^z|$$

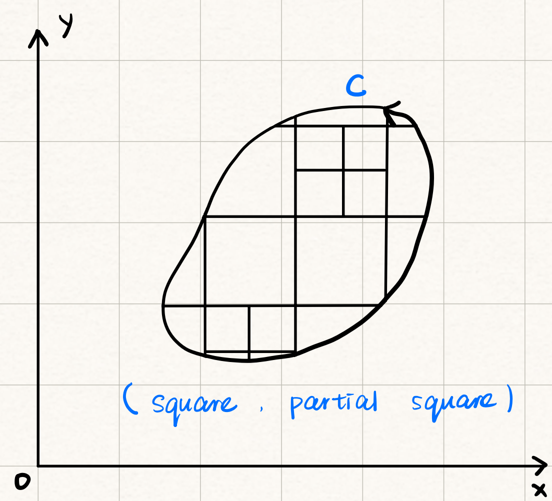
$$\overset{z=x+iy}{=} \max_{z \in C} |e^{x+iy}| = \max_{z \in C} |e^x| \cdot \underbrace{|e^{iy}|}_{= 1}$$

$$= \max_{z \in C} |e^x|$$

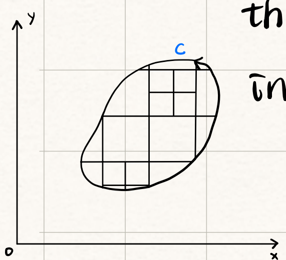
$$\overset{x \in [-1,1]}{=} e .$$

$$\Rightarrow \left| \int_c \frac{e^z}{z}\, dz \right| \le \pi e .$$





Lemma : Let f be holomorphic throughout a closed region R which is bounded by C. Then for any positive number ε, the region R can be covered with a finite number of squares and partial squares K_j for $j = 1, \cdots, n$, such that in each K_j, there is a fixed point $z_j \in K_j$ for which the inequality



$$\left| \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \right| < \varepsilon$$

is satisfied by all $z \in K_j$ other than z_j.

On the j-th square or partial square (K_j), define a function $\delta_j : K_j \to \mathbb{C}$

$$\delta_j(z) = \begin{cases} 0 & \text{if } z = z_j \\ \dfrac{f(z) - f(z_j)}{z - z_j} - f'(z_j) & \text{if } z \neq z_j \end{cases}.$$

According to the Lemma,

$|\delta_j(z)| < \varepsilon$ for z on the jth square / partial K_j

Notice that $\delta_j(z)$ is continuous on K_j.

$$\lim_{z \to z_j} \delta_j(z) = f'(z_j) - f'(z_j) = 0$$

Let C_j be the positively oriented boundries of K_j, $j = 1, \dots, n$.

By the expression of $\delta_j(z)$.

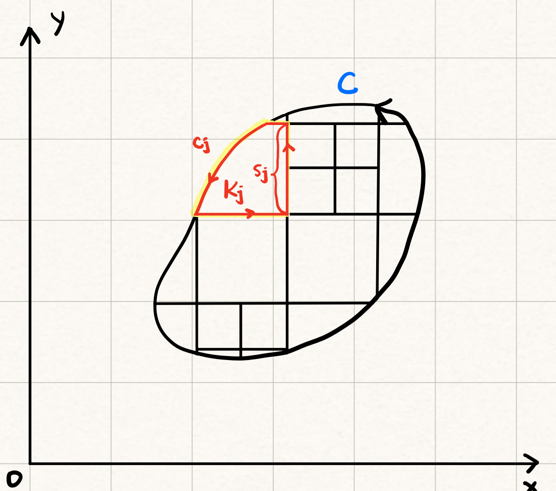
$$f(z) = \underbrace{f(z_j) - z_j f'(z_j)}_{\text{constant}} + f'(z_j) z + (z - z_j)\delta_j(z).$$

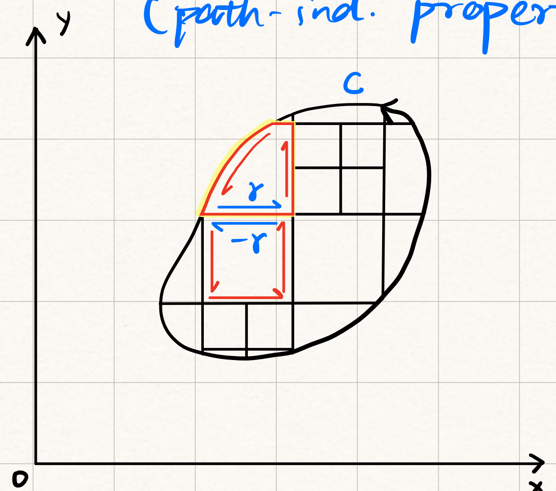
Integrate over C_j on both sides.

$$\int_{C_j} f(z)\,dz = [f(z_j) - z_j f'(z_j)] \cancel{\int_{C_j} dz} + f'(z_j) \cancel{\int_{C_j} z\,dz} + \int_{C_j} (z - z_j)\delta_j(z)\,dz.$$

We have $\int_{C_j} dz = 0$, $\int_{C_j} z\,dz = 0$.

since 1 and z have antiderivative. (path-ind. property).





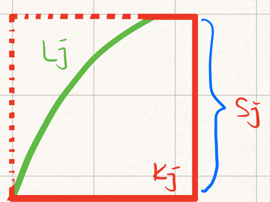
$$\int_\gamma = -\int_{-\gamma}$$

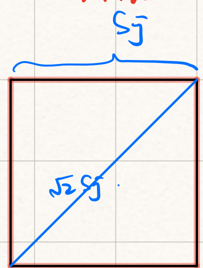
$$\int_{C_j} f(z)\,dz = \int_{C_j} (z-z_j)\,\delta_j(z)\,dz$$

Observe that $\sum_{j=1}^{n} \int_{C_j} f(z)\,dz = \int_C f(z)\,dz$.

$$\Rightarrow \int_C f(z)\,dz = \sum_j \int_{C_j} (z-z_j)\,\delta_j(z)\,dz$$

$$\Rightarrow \left|\int_C f(z)\,dz\right| \le \sum_j \left|\int_{C_j} (z-z_j)\,\delta_j(z)\,dz\right| \overset{\text{Goal.}}{\underset{\text{Aim.}}{\le}} h\cdot\varepsilon.$$





$$|z-z_j| \le \sqrt{2}\,S_j.$$

$$|(z-z_j)\,\delta_j(z)| = |z-z_j|\cdot|\delta_j(z)| < \sqrt{2}\,S_j\,\varepsilon$$

If K_j is a square, length of C_j equal to $4S_j$.

$$\Rightarrow \left|\int_{C_j} (z-z_j)\,\delta_j(z)\,dz\right| < \sqrt{2}\,S_j\,\varepsilon\cdot 4S_j = 4\sqrt{2}\,\varepsilon\,S_j^2 =: 4\sqrt{2}\,\varepsilon\,A_j.$$

If K_j is a partial square, then ... $< 4S_j + L_j$

$$\Rightarrow \left|\int_{C_j} (z-z_j)\,\delta_j(z)\,dz\right| < \sqrt{2}\,S_j\,\varepsilon\,(4S_j + L_j) < 4\sqrt{2}\,\varepsilon\,A_j + \sqrt{2}\,S\,\varepsilon\,L_j$$

Finally: $\left| \int_C f(z)\,dz \right| < 4\sqrt{2}\,\varepsilon \sum_j A_j$

$$+ \sqrt{2}\, S \; \varepsilon \sum_j L_j$$

$$= (4\sqrt{2}\, S^2 + \sqrt{2}\, S L)\, \varepsilon.$$

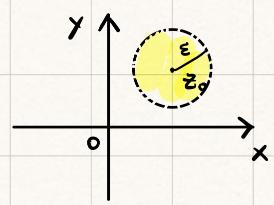
$$\Rightarrow \int_C f(z)\,dz = 0.$$

Consider the complex plane $\mathbb{C}$ as the whole space, $S \subseteq \mathbb{C}$ is a subset of $\mathbb{C}$. $\varepsilon > 0$ is a given positive real number which is arbitrary.

(given point)

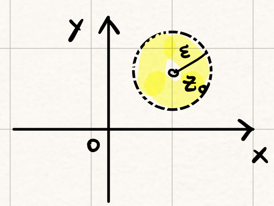
Def: An ε-neighborhood (ε-nbhd) of z_0 is the

(i) set $\{ z \in \mathbb{C} : |z - z_0| < \varepsilon \}$



(ii) A deleted ε-nbhd of z_0 is the set

$$\{ z \in \mathbb{C} : 0 < |z - z_0| < \varepsilon \}$$



(iii) A point z_0 is an accumulation point of a set S if each deleted nbhd of z_0 contains at least one point of S.

(iv) A point z_0 is an interior point of S if there is some nbhd of z_0 contains only point of S.

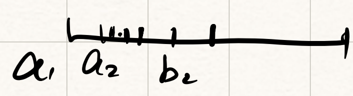
(v) A point z_0 is an exterior point of S if there is some nbhd of z_0 contains no point of S.

(vi) A point z_0 is a boundary point of S if z_0 is neither interior nor exterior point.

(vii) A set is closed if it contains all of its boundary point.

Property: A set S is closed if and only if it contains all of its accumulation points.

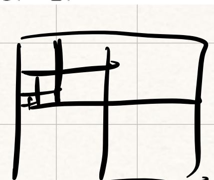
$\{[a_n, b_n]\}_{n \in \mathbb{N}}$



Section 53.

8. *Nested Intervals*. An infinite sequence of closed intervals $a_n \le x \le b_n$ $(n = 0, 1, 2, \ldots)$ is formed in the following way. The interval $a_1 \le x \le b_1$ is either the left-hand or right-hand half of the first interval $a_0 \le x \le b_0$, and the interval $a_2 \le x \le b_2$ is then one of the two halves of $a_1 \le x \le b_1$, etc. Prove that there is a point x_0 which belongs to every one of the closed intervals $a_n \le x \le b_n$.

9. *Nested Squares*. A square $\sigma_0 : a_0 \le x \le b_0, c_0 \le y \le d_0$ is divided into four equal squares by line segments parallel to the coordinate axes. One of those four smaller squares $\sigma_1 : a_1 \le x \le b_1, c_1 \le y \le d_1$ is selected according to some rule. It, in turn, is divided into four equal squares one of which, called σ_2, is selected, etc. (see Sec. 49). Prove that there is a point (x_0, y_0) which belongs to each of the closed regions of the infinite sequence $\sigma_0, \sigma_1, \sigma_2, \ldots$.



(x_0, y_0)

$(\sigma_i)_{i \in \mathbb{N}}$.

Proof of Lemma

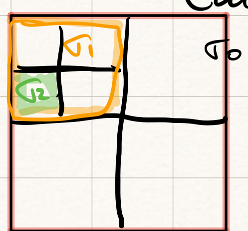
idea: Suppose such point $\nexists$ if we subdivide a finite number of times

Denote σ_0, find nested infinite sequence

$$\sigma_0 \supset \sigma_1 \supset \cdots \supset \sigma_k \supset \cdots$$

with no appropriate point z_j.

Then by ...., there is a point z_0 common to each σ_j, $j \in \mathbb{N}$.



Each of σ_j contains pts in R other than z_0.

$\sigma_j \downarrow$, then $\forall \delta > 0$ δ-nbhd of z_0 must contain some σ_N.

$\Rightarrow$ δ-nbhd contains pts of R distinct from z_0.

$\Rightarrow$ z_0 is a ac. pt. of R.

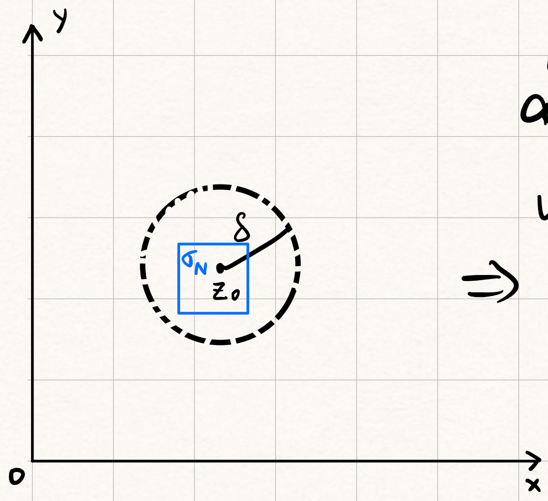
$\Rightarrow$ $z_0 \in R$ since R is closed in $\mathbb{C}$.

Then $f'(z_0)$ $\exists$ since f holo. in R.

For $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon.$$

for all $z \in \{ z \in \mathbb{C} : |z - z_0| < \delta \}$.



By any δ-nbhd contains a σ_N square.

when σ_N small enough.

$\Rightarrow$ z_0 is the "z_j" in σ_N for the Lemma,

$\lightning$

$\square$.